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Two-dimensional position-dependent massive particles in the presence of magnetic fields

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Abstract
In this work, we discuss the problem of a particle with a position-dependent mass interacting with a two-dimensional potential well with finite depth, as well as under the influence of a uniform magnetic field. The ordering ambiguity is taken in account, and we obtain the exact wavefunctions and energies for a complete set of eigenstates. It is shown that, even considering a finite-depth potential well, the system retains an infinite set of quantum states.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

After the advent of the modern fabrication of nano-devices such as quantum dots, wires and wells, a naturally renewed interest in the exact solution of two-dimensional confined systems has appeared [1]. One of the possible ways used to try to keep exact solvability on this track is to consider the usual exactly solvable potentials under infinite quantum wells, or, in other words, considering those problems under convenient Dirichlet boundary conditions [2, 3]. Furthermore, in order to take into account the spatial variation of the semiconductor type, some effective Hamiltonians proposed include a spatially dependent mass for the carrier [4–10]. As an important consequence, in general, this kind of system becomes ambiguous at the quantum level. However, the problem of ordering ambiguity is one of the long-standing unsolved questions of quantum mechanics. This matter has attracted the attention of some of the founders of quantum mechanics. Born and Jordan, Weyl, Dirac and von Newmann worked on this problem, as can be verified from the excellent review by Shewell [11]. This is a hard problem which has advanced very few along the last decades. Notwithstanding, as a consequence of its importance for the modeling of some experimental situations like

impurities in crystals [12–14], the dependence of nuclear forces on the relative velocity of the two nucleons [15, 16] and, more recently, the study of semiconductor heterostructures [17, 18], the interest in such kind of systems never vanished. Some time ago, it was discussed the exact solvability of some classes of one-dimensional Hamiltonians with ordering ambiguity [19] and, after that, a dependence of the ordering ambiguity with the representation chosen to study it [21]. In fact, the study of systems with spatially dependent mass is presenting a growing interest along the last few years [19–46]. However, as far as we know, the majority part of the works dedicated to this research, deals with one-dimensional systems. Although, there are physical systems like those where a magnetic field [48–51] is present, which lead naturally to the necessity of a two-dimensional analysis. In this work, we intend to partially fill this gap.

Here, we will address the position-dependent mass (PDM) type of systems in two spatial dimensions by using Cartesian coordinates. We will introduce a very interesting system which, as we are going to see below in the manuscript, is able to confine an arbitrary number of particles in a finite-size deep well. Furthermore, we will also implement the interaction of the particles confined in that mentioned potential with an uniform magnetic field.

2. Effective Schrödinger equation in two-dimensional Cartesian coordinates

In this section, we discuss the case of a system with a two-dimensional PDM under the influence of a potential of a smooth and finite-depth well. The exact solution of the corresponding Schrödinger equation is obtained, and the $SU(2)$ coherent states are constructed. The problem of ordering ambiguity is taken into account and, as a consequence, we need to choose a particular ordering or, equivalently, the form of the potential in order to get the mentioned exact solution.

First of all, we begin by using the ordering defined by von Roos [8, 19]. Besides, for the Hamiltonian operator, which in one-dimensional space is written as

$$\hat{H} = \tfrac{1}{4}(M^{\alpha}\hat{p}M^{\beta}\hat{p}M^{\gamma} + M^{\gamma}\hat{p}M^{\beta}\hat{p}M^{\alpha}) + V(x), \tag{1}$$

where $\hat{p}$ is the momentum operator and $M = M(x)$ is the position-dependent effective mass. α, β and γ are arbitrary ordering parameters which must obey the relation

$$\alpha + \beta + \gamma = -1, \tag{2}$$

in order to grant the correct classical limit.

Now, by using the canonical commutation relations, and putting the momentum operators at the right-hand side, one gets the following effective Hamiltonian operator:

$$M^{\gamma}\hat{p}M^{\beta}\hat{p}M^{\alpha} = \frac{\hat{p}^2}{M} - \mathrm{i}\hbar(\beta+2\alpha)\frac{M'}{M^2}\hat{p} - \hbar^2\alpha(\beta+\alpha-1)\frac{(M')^2}{M^3} - \hbar^2\alpha\frac{M''}{M^2}. \tag{3}$$

By using the relation $\alpha+\beta+\gamma = -1$, we eliminate the parameter $\beta = -(\alpha+\gamma+1)$, and arrive at the following effective Hamiltonian operator [19]:

$$H = \frac{1}{2M}\hat{p}^2 + \frac{\mathrm{i}\hbar}{2}\frac{M'}{M^2}\hat{p} + U(\alpha,\gamma,x) + V(x). \tag{4}$$

The effective potential $U(\alpha,\gamma,x)$ is, then, written as

$$U(\alpha,\gamma,x) = -\frac{\hbar^2}{4M^3}\left[(\alpha+\gamma)M\left(\frac{\partial^2 M}{\partial x^2}\right) - 2(\alpha+\gamma+\alpha\gamma)\left(\frac{\partial M}{\partial x}\right)^2\right]. \tag{5}$$

In this case, the corresponding differential equation can be written as

$$-\frac{\hbar^2}{2M(x)}\frac{\mathrm{d}^2x}{\mathrm{d}x^2} + \frac{\hbar^2}{2}\left[\frac{\mathrm{d}M/\mathrm{d}x}{M^2}\right]\frac{\mathrm{d}\psi}{\mathrm{d}x} + [V(x) + U(\alpha,\gamma,x) - E]\,\psi = 0. \tag{6}$$

Now, we can extend the above calculations for the case of a set of two-dimensional Cartesian coordinates, where we deal with a mass like $M = M(x, y)$. In that case, one can obtain the following expression for the effective Hamiltonian operator:

$$H = \frac{1}{2M}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{\mathrm{i}\hbar}{2}\left(\frac{\frac{\partial M}{\partial x}\hat{p}_x + \frac{\partial M}{\partial y}\hat{p}_y}{M^2}\right) + U(\alpha, \gamma, x, y) + V(x, y), \tag{7}$$

where $V(x, y)$ is the original potential of the system, and $U(\alpha, \gamma, x, y)$ is the effective one, which can be expressed as

$$U(\alpha, \gamma, x, y) = -\frac{\hbar^2}{4M}\left\{(\alpha + \gamma)\frac{M_{xx} + M_{yy}}{M} - 2(\alpha + \gamma + \alpha\gamma)\left[\left(\frac{M_x}{M}\right)^2 + \left(\frac{M_y}{M}\right)^2\right]\right\}, \tag{8}$$

with $M_x \equiv \frac{\partial M}{\partial x}$ and $M_y \equiv \frac{\partial M}{\partial y}$. Thus, the corresponding effective operator can be rewritten as follows:

$$H = \frac{1}{2M}\overrightarrow{p}^2 + \frac{\mathrm{i}\hbar}{2}\frac{1}{M^2}\overrightarrow{\nabla} M \cdot \overrightarrow{p} + U(\alpha, \gamma, x, y) + V(x, y), \tag{9}$$

where

$$U(\alpha, \gamma, x, y) \equiv -\frac{\hbar^2}{4M}\left[(\alpha + \gamma)\frac{\nabla^2 M}{M} - 2(\alpha + \gamma + \alpha\gamma)\left(\frac{\overrightarrow{\nabla} M}{M}\right)^2\right]. \tag{10}$$

Have we started with a typical Schrödinger equation

$$-\frac{\hbar^2}{2M(x, y)}\nabla^2\chi + V_{\mathrm{eff}}(x, y)\chi = E\chi, \tag{11}$$

and performed the substitution

$$\chi(x, y) = \mathrm{e}^{\sigma(x,y)}\psi(x, y), \tag{12}$$

we would obtain the following transformed equation:

$$-\frac{\hbar^2}{2M(x, y)}\nabla^2\psi - \frac{\hbar^2}{M(x, y)}[(\overrightarrow{\nabla}\sigma)\cdot\overrightarrow{\nabla}\psi] + \left\{V(x, y) - \frac{\hbar^2}{2M(x, y)}[\nabla^2\sigma + (\overrightarrow{\nabla}\sigma)^2]\right\}\psi = E\psi, \tag{13}$$

which corresponds to a Hamiltonian operator given by

$$H = \frac{1}{2M(x, y)}\overrightarrow{p}^2 - \frac{\hbar^2}{M(x, y)}\frac{\mathrm{i}}{\hbar}(\overrightarrow{\nabla}\sigma)\cdot\overrightarrow{p} + V - \frac{\hbar^2}{2M(x, y)}[\nabla^2\sigma + (\overrightarrow{\nabla}\sigma)^2]. \tag{14}$$

Now, requiring that this last Hamiltonian matches with the one appearing in equation (9), we get the following condition:

$$-\frac{\hbar^2}{M}\frac{\mathrm{i}}{\hbar}\overrightarrow{\nabla}\sigma\cdot\overrightarrow{p} = \frac{\mathrm{i}\hbar}{2}\frac{1}{M^2}\overrightarrow{\nabla} M\cdot\overrightarrow{p}, \tag{15}$$

whose solution is given by

$$\sigma = \ln\left(M^{-\frac{1}{2}}\right). \tag{16}$$

Thus, one finishes finally with

$$H = \frac{1}{2M}\overrightarrow{p}^2 + \frac{\mathrm{i}\hbar}{2M}\frac{\overrightarrow{\nabla} M}{M}\cdot\overrightarrow{p} + \left\{V - \frac{\hbar^2}{4M}\left[\frac{3}{2}\left(\frac{\overrightarrow{\nabla} M}{M}\right)^2 - \frac{\nabla^2 M}{M}\right]\right\}, \tag{17}$$

and the wavefunction is re-scaled as

$$\psi = M^{\frac{1}{2}}\chi. \tag{18}$$

Thus, we can conclude that, starting from the effective Hamiltonian (7), one can reach to the effective Schrödinger equation presented in equation (11), where the effective potential is

$$V_{\rm eff}(x,y) = V(x,y) + \frac{\hbar^2}{4M}\left[2\left(\alpha+\gamma+\alpha\gamma+\frac{3}{4}\right)\left(\frac{\overrightarrow{\nabla}M}{M}\right)^2 - (\alpha+\gamma+1)\frac{\nabla^2 M}{M}\right]. \tag{19}$$

At this point a comment about the ordering ambiguity is in order. Comparing the above result with some other in the literature [5, 23, 37, 40] for the one-dimensional case, one can verify that, apart from the fact that those authors eliminate the parameter γ instead of β by using the equation (2), and that they work with $\hbar = 2\ m_0 = 1$, there is a difference in the effective potential. This is due to the presence of the terms $\frac{\hbar^2}{4M}\left[2\left(\frac{3}{4}\right)\left(\frac{\overrightarrow{\nabla}M}{M}\right)^2 - \frac{\nabla^2 M}{M}\right]$. This happens because, here we work with a kinetic term like $\frac{1}{M}\nabla^2$, instead of $\vec{\nabla}\left(\frac{1}{M}\right)\vec{\nabla}$ which is used by the quoted works. So, there is no contradiction between this result with the one obtained in those references.

In order to work with an equivalent system with constant mass, we can write equation (11) as

$$-\frac{\hbar^2}{2}\nabla^2\chi + U_{\rm eff}\chi = \xi\chi, \tag{20}$$

where

$$U_{\rm eff} - \xi = M(x,y)V(x,y) + \frac{\hbar^2}{4}\left[2\left(\alpha+\gamma+\alpha\gamma+\frac{3}{4}\right)\left(\frac{\overrightarrow{\nabla}M}{M}\right)^2 - (\alpha+\gamma+1)\frac{\nabla^2 M}{M}\right] - EM(x,y), \tag{21}$$

with ξ constant.

Let us now analyze an example which can be exactly solved. Having the exact solvability in mind, we choose to work in this example with a case where one deals with a harmonically increasing mass like

$$M(x,y) = M_0\left[1+\frac{g}{2}(x^2+y^2)\right], \tag{22}$$

leading us to

$$U_{\rm eff} - \xi = M_0\left[1+\frac{g}{2}(x^2+y^2)\right]V(x,y) - EM_0\left[1+\frac{g}{2}(x^2+y^2)\right] + \frac{\hbar^2}{4}\left\{2\left(\alpha+\gamma+\alpha\gamma+\frac{3}{4}\right)\frac{g^2}{M_0^2}\left[\frac{x^2+y^2}{1+\frac{g}{2}(x^2+y^2)}\right] - (\alpha+\gamma+1)\frac{2g}{M_0\left[1+\frac{g}{2}(x^2+y^2)\right]}\right\}. \tag{23}$$

In fact, in order to guarantee that the example is exactly solvable, one should choose an ordering dependent potential or, more easily, restrict the study to the case with the following ordering:

$$\alpha+\gamma+1 = 0, \alpha+\gamma+\alpha\gamma+\tfrac{3}{4} = 0, \tag{24}$$

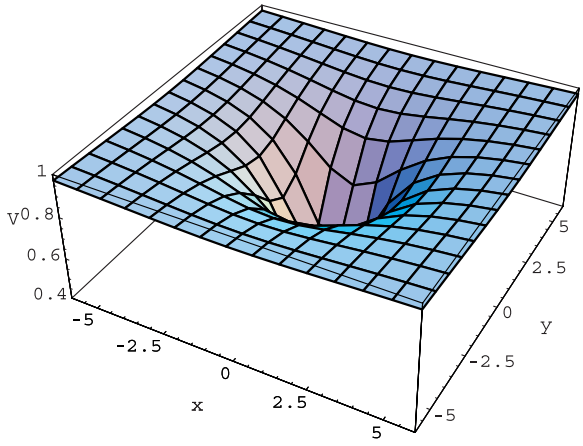

Figure 1. $V(x, y) = \frac{\omega^2 x^2 + \omega^2 y^2}{2\left[1+\frac{g}{2}(x^2+y^2)\right]}$ with $\omega = 1$ and $g = 1$.

whose solution is given by

$$\alpha = -\tfrac{1}{2}, \qquad \gamma = -\tfrac{1}{2}, \qquad \beta = 0, \tag{25}$$

in such a way that we arrive at an ordering where

$$\frac{\overrightarrow{p}^2}{2M} = \frac{1}{2}\left(\frac{1}{\sqrt{M}}\overrightarrow{p}\right)\cdot\left(\overrightarrow{p}\,\frac{1}{\sqrt{M}}\right) = \frac{1}{2}\frac{1}{\sqrt{M}}(\overrightarrow{p})^2\frac{1}{\sqrt{M}}. \tag{26}$$

In this case, we can write a much simpler effective potential

$$U_{\text{eff}} - \xi = M_0\left[1 + \frac{g}{2}(x^2+y^2)\right]V(x, y) - EM_0 - \frac{EM_0 g}{2}(x^2+y^2). \tag{27}$$

However, the example will be complete only when we have defined the potential under which the PDM particle is moving. Here we choose an anisotropic one, representing a kind of finite-depth well, as can be seen from figure 1 in the isotropic situation ($\omega = 1$ and $g = 1$). In general we have

$$V(x, y) = \frac{\omega_1^2 x^2 + \omega_2^2 y^2}{2\left[1 + \frac{g}{2}(x^2+y^2)\right]}, \tag{28}$$

which lead us to

$$\begin{aligned} U_{\text{eff}} - \xi &= \frac{1}{2}M_0\left(\omega_1^2 x^2 + \omega_2^2 y^2\right) - EM_0 - \frac{EM_0}{2}g(x^2+y^2) \\ &= \frac{1}{2}M_0\left[\left(\omega_1^2 - Eg\right)x^2 + \left(\omega_2^2 - Eg\right)y^2\right] - EM_0, \end{aligned} \tag{29}$$

then, if we define $\xi \equiv EM_0$, we get for the energy spectrum the relation

$$EM_0 = \hbar\sqrt{\left(\omega_1^2 - Eg\right)M_0}\left(n + \tfrac{1}{2}\right) + \hbar\sqrt{\left(\omega_2^2 - Eg\right)M_0}\left(m + \tfrac{1}{2}\right), \tag{30}$$

where $n, m = 0, 1, 2, \ldots$, and we must impose that $Eg < \omega_1^2$ and $Eg < \omega_2^2$, in order to keep the energy spectrum real and the wavefunctions normalizable. The analytical expression for the energy spectrum in this case is rather cumbersome. However, the numerical solution can be easily done and we present an example of this case in figure 2. There, we plotted the value of the energy levels as a function of the quantum number n, with $m = 0, \ldots, 5$, for the case where $\omega_2 = 2\omega_1$ and $\omega_1 = 10$. Note that, as the quantum number increases, the energy

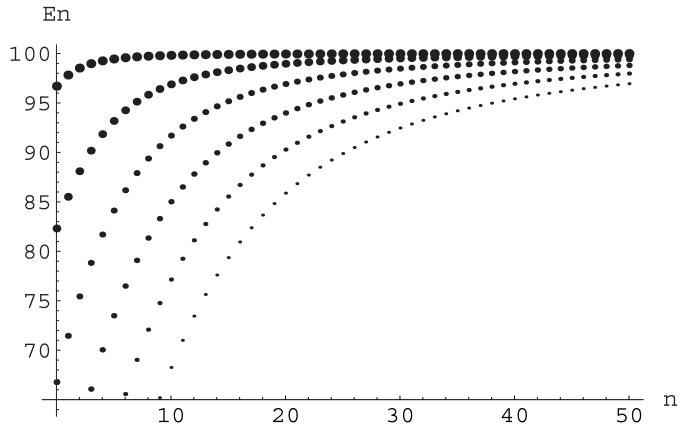

Figure 2. Typical energy behavior for increasing quantum number n. Here, $\omega_2 = 2\omega_1$, $\omega_1 = 10$, $g = M_0 = \hbar = 1$ and $m = 0, 1, 2, 3, 4, 5$ (the points become thicker as m is increasing).

eigenvalue approaches a limit value. Furthermore, one can observe that there is a limit in the quantum number associated with the higher frequency (ω_2 in this example). In fact, one can verify that, since the energy goes asymptotically in n to $\frac{\omega_1^2}{g}$, this maximum value of the quantum number m is given by

$$m_{\max} = I_{nt}\left[\frac{M_0\omega_1^2}{\hbar g\sqrt{M_0\left(\omega_2^2-\omega_1^2\right)}}\right], \tag{31}$$

where $I_{nt}[\cdot]$ stands for the integer part of the quantity inside the square brackets ($m_{\max} = 5$ in this example). This can be understood because the effective oscillator frequencies depend on the energy, and after this maximum quantum number, the frequency would become imaginary, rendering a non-normalizable wavefunction. Moreover, when the frequencies become equal in the isotropic limit, both quantum numbers are unlimited and the system, as expected, presents a degeneracy. This can be noted also from the fact that, when $E \to \omega^2/g$, the term $\sqrt{\omega^2 - Eg}$ becomes asymptotically small, so that the quantum number can increase arbitrarily, in such a way that the product $\sqrt{\omega^2 - Eg}\,(n + 1/2)$ stays limited as required. This behavior will be confirmed analytically below, when we will be concerned with the isotropic case.

In this last part of the section, we are going to construct $SU(2)$ coherent states, as studied by Chen *et al* [52, 53]. Now we restrict ourselves to the isotropic case where $\omega_1 = \omega_2 = \omega$. In that case, the energy spectrum must obey the following simpler equation:

$$M_0E^2 + Q_{nm}gE - Q_{nm}\omega^2 = 0, \tag{32}$$

where $Q_{nm} = \hbar^2(n + m + 1)^2$. Then, we obtain finally that

$$E_{nm} = \frac{1}{2M_0}\left(-Q_{nm}g + \sqrt{Q_{nm}^2g^2 + 4M_0\omega^2Q_{nm}}\right). \tag{33}$$

Analyzing this energy spectrum, one can observe that there are infinitely many allowed bound states, showing that the quadratically growing mass, conspires to permit that a finite-depth well becomes able to sustain an arbitrarily high number of quantum bound states, which is usually incompatible with this kind of potential in the constant mass case. This can be verified by noting that the limit when the quantum number goes to infinity is

$$\lim_{N\to\infty} E_N \to \frac{\omega^2}{g}, \tag{34}$$

where $N \equiv n + m$. This limit is precisely the depth of the potential which is under analysis. Thus, we conclude that an infinite set of quantum states is allowed in this arrangement.

Finally, we finish this section by constructing the so-called $SU(2)$ coherent states [52, 53], since they present a very interesting feature. Those states usually have the highest probability of finding the particles precisely over the classical trajectory. Besides, these are stationary states, as well as the usual eigenfunctions for the harmonic oscillators. Then, we decided to present them here, instead of the usual eigenfunctions. In fact, as we are going to see, these states lead to the conclusion that a behavior which is very similar to that of the usual two-dimensional harmonic oscillator takes place, and the line where the probability reach its maximum is a circle also in the present situation. This is a nontrivial feature, because similar constructions performed in the case of the Wigner distribution of one-dimensional PDM systems, presented a different behavior when compared to the original harmonic oscillator distribution [54]. The $SU(2)$ coherent states can be written by using the definition [52, 53]

$$\Phi(x, y, \tau) = \frac{1}{(1 + |\tau|^2)^{\frac{L}{2}}} \sum_{K=0}^{L} \binom{L}{K}^{1/2} \tau^K \psi_{nm}(x, y), \tag{35}$$

with $K = 0, 1, 2, \ldots, L$ and p and q are integer numbers. The complex parameter $\tau = A\, e^{i\phi}$, where $\phi = \frac{\pi}{2}$, written in terms of polar coordinates is used to make the connection with the classical trajectory. It can be observed in equation (35) that the $SU(2)$ coherent state is a superposition of the degenerate energy states. Because the energy eigenfunctions can be obtained from equation (18), one gets

$$\psi_{nm}(x, y) = \frac{1}{\sqrt{2^{(m+n+1)}\pi n! m!}} \frac{M(x, y)^{1/2}}{\sqrt{R^2}} H_m\left(\frac{\sqrt{2}x}{R}\right) H_n\left(\frac{\sqrt{2}y}{R}\right) \exp\left[-\left(\frac{x}{R}\right)^2 - \left(\frac{y}{R}\right)^2\right], \tag{36}$$

where $R = \sqrt{2\hbar/(M_0\sqrt{(\omega^2 - Eg)M_0})}$.

Figure 3 shows the probability density for the case where $L = 20$ and $p = q = 1$, as the one used by Chen and Huang in [52]. There, it was used $\hbar = M_0 = A = \omega = g = 1$, and the energy is always lesser than the frequency in equation (30). The right-hand side plot in figure 3 is just a projection of the left-hand one.

3. PDM in a magnetic field

In this section, we will analyze the effect of a uniform magnetic field over the behavior of a charged PDM particle bounded by the potential introduced in the previous section. For this, we begin by doing a discussion about this problem in general, then we consider the particular case of the uniform magnetic field in a Coulomb gauge. In this situation, the classical Hamiltonian is given by

$$\begin{aligned} H &= \frac{1}{2M(x, y)}(\vec{p} - e\,\vec{A})^2 + V(x, y) \\ &= \frac{1}{2M}\vec{p}^{\,2} - \frac{e}{2M}(\vec{A}\cdot\vec{p} + \vec{p}\cdot\vec{A}) + \frac{e^2}{2M}\vec{A}^2 + V(x, y), \end{aligned} \tag{37}$$

where e is the electric charge and $\vec{A}(x, y)$ is the vector potential. Defining

$$\vec{\tilde{A}} \equiv \frac{\vec{A}}{M}, \tag{38}$$

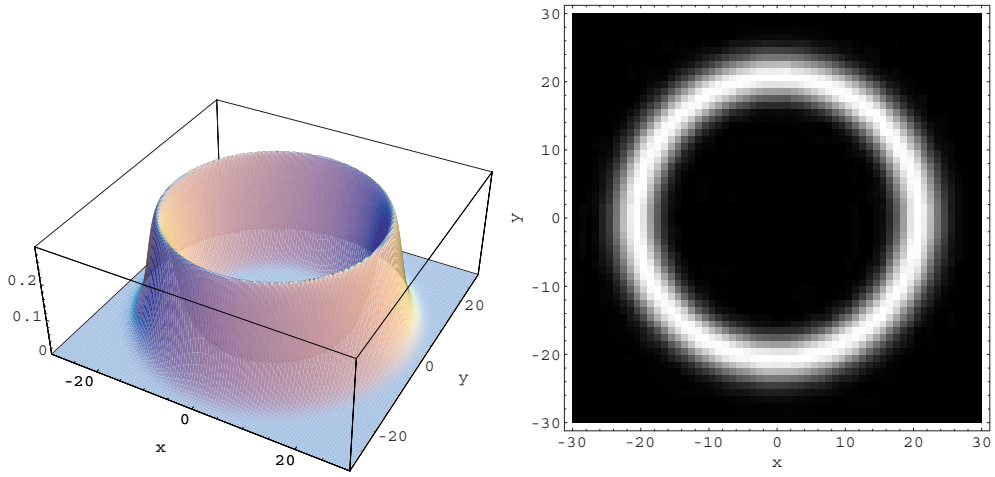

Figure 3. Probability density when $\omega = 1$.

we get the following classical interaction Hamiltonian:

$$H = \frac{1}{2M}\vec{p}^{\,2} - \frac{e}{2}(\vec{A}\cdot\vec{p} + \vec{p}\cdot\vec{A}) + \frac{e^2}{2}M\vec{A}^{2} + V(x, y). \tag{39}$$

In the previous section, we discussed the ordering of the quantum operator which corresponds to the kinematical term in the Hamiltonian (39). Thus, we already know its expression, which shall be the same even when we take into account the electromagnetic interaction. However, we need to take into account the ordering of the operator linear in $\vec{p}$, which is such that

$$\vec{A}\cdot\vec{p} = \tilde{A}_x p_x + \tilde{A}_y p_y, \tag{40}$$

thus, it is decoupled into two one-dimensional operators of the type $f(x)p_x$. Now, by using the analysis of this term, as done in [19, 21], one can see that the ordering of a one-dimensional term like

$$\hat{O} = \tfrac{1}{2}[f(x)^{\alpha}\hat{p}f(x)^{\beta} + f(x)^{\beta}\hat{p}f(x)^{\alpha}] \tag{41}$$

can be done by using the commutation relation between the momentum and the spatial variable in (41). One shall also use the constraint among the ordering parameters $\alpha + \beta = 1$, such that one arrives at the following expression:

$$\hat{O} = f(x)\hat{p} - \frac{i\hbar}{2}\frac{df(x)}{dx}. \tag{42}$$

Thus, we can see that, in the two-dimensional Cartesian coordinates, the ordering for the operator which is linear in $\vec{p}$ in equation (39), acquires the form

$$\hat{O} = \tfrac{1}{2}\big(\tilde{A}_x^{\alpha}\hat{p}_x\tilde{A}_x^{\beta} + \tilde{A}_y^{\alpha}\hat{p}_y\tilde{A}_y^{\beta} + \tilde{A}_x^{\beta}\hat{p}_x\tilde{A}_x^{\alpha} + \tilde{A}_y^{\beta}\hat{p}_y\tilde{A}_y^{\alpha}\big), \tag{43}$$

and we finish with

$$\hat{O} = \frac{\hbar}{i}\vec{A}\cdot\vec{\nabla} - \frac{i\hbar}{2}\vec{\nabla}\cdot\vec{A}. \tag{44}$$

Now, substituting equation (38) in the above equation, one is lead to

$$\hat{O} = \frac{\hbar}{i}\frac{\vec{A}}{M}\cdot\vec{\nabla} - \frac{i\hbar}{2M}\vec{\nabla}\cdot\vec{A} + \frac{i\hbar}{2M^2}\vec{\nabla}M\cdot\vec{A}. \tag{45}$$

Here, we choose to work with the Coulomb gauge: $\overrightarrow{\nabla} \cdot \overrightarrow{A} = 0$. So, we are left simply with

$$\hat{O} = \frac{\hbar}{\mathrm{i}} \frac{\overrightarrow{A}}{M} \cdot \overrightarrow{\nabla} + \frac{\mathrm{i}\hbar}{2M^2} \overrightarrow{\nabla} M \cdot \overrightarrow{A}. \tag{46}$$

Let us suppose now that we have a uniform magnetic field in the z-direction, $\overrightarrow{B} = B_0 \hat{z}$. It can be obtained when use is made of the vector potential in the so-called symmetric gauge form, where

$$\overrightarrow{A} = \frac{B_0}{2}(-y\hat{i} + x\hat{j}), \tag{47}$$

and

$$\overrightarrow{\nabla} M \cdot \overrightarrow{A} = \frac{\partial M}{\partial x} A_x + \frac{\partial M}{\partial y} A_y = \frac{B_0}{2} \left(\frac{\partial M}{\partial x}(-y) + \frac{\partial M}{\partial y} x \right). \tag{48}$$

Here, we will work with the case where the mass presents the same spatial dependence which we studied in the last section,

$$M(x, y) = M_0 \left[1 + \frac{g}{2}(x^2 + y^2) \right], \tag{49}$$

and this implies that $(\overrightarrow{\nabla} M) \cdot \overrightarrow{A} = M_0[-y(gx) + x(gy)] = 0$. Then, we have

$$\hat{O} = \frac{\hbar}{\mathrm{i}} \frac{\overrightarrow{A}}{M} \cdot \overrightarrow{\nabla} = -\frac{\mathrm{i}\hbar B_0}{M}(x\partial_y - y\partial_x).$$

Let us now see what is the effect of this operator when it is applied into a function like $(M^{1/2}\chi)$. Then we get

$$(x\partial_y - y\partial_x)(M^{1/2}\chi) = \frac{xM_y - yM_x}{2M^{1/2}}\chi + M^{1/2}(x\partial_y - y\partial_x)\chi = M^{1/2}(x\partial_y - y\partial_x)\chi.$$

This allow us to conclude that, starting from the classical Hamiltonian (37), and using the PDM we have chosen to work with here, one is left with the following effective Schrödinger equation:

$$-\frac{\hbar^2}{2M(x, y)}\nabla^2\psi + \frac{\hbar^2}{2M(x, y)^2}[\overrightarrow{\nabla} M(x, y)] \cdot \overrightarrow{\nabla}\psi + V_{\mathrm{eff}}(\alpha, \gamma, x, y)\psi + \frac{\mathrm{i}\hbar B_0}{M(x, y)}(x\partial_y - y\partial_x)\psi + \frac{\mathrm{e}^2 B_0^2 (x^2 + y^2)}{8M(x, y)}\psi = E\psi, \tag{50}$$

and by using the relation (18) we finish with

$$-\frac{\hbar^2}{2M(x, y)}\nabla^2\chi + \frac{\mathrm{i}\hbar B_0 e}{2M(x, y)}(x\partial_y - y\partial_x)\chi + V_{\mathrm{eff}}(\alpha, \gamma, x, y)\chi = E\chi, \tag{51}$$

or, equivalently, with the constant mass effective Schrödinger equation

$$-\frac{\hbar^2}{2}\nabla^2\chi + \frac{\mathrm{i}\hbar B_0 e}{2}(x\partial_y - y\partial_x)\chi + U_{\mathrm{eff}}(\alpha, \gamma, x, y)\chi = \xi\chi, \tag{52}$$

where the effective potential is defined in such a way that one has

$$U_{\mathrm{eff}} - \xi = V(x, y)M(x, y) + \frac{\hbar^2}{4}\left[2\left(\alpha + \gamma + \alpha\gamma + \frac{3}{4}\right)\left(\frac{\overrightarrow{\nabla} M}{M}\right)^2 - (\alpha + \gamma + 1)\frac{\nabla^2 M}{M} \right] + \frac{\mathrm{e}^2 B_0^2}{8}(x^2 + y^2) - EM(x, y). \tag{53}$$

Now, aiming to work with an exactly solvable case, besides to choose the special form for the PDM which appears in (49), we use the very same ordering used in the case without magnetic interaction $\alpha = -\frac{1}{2}$, $\gamma = -\frac{1}{2}$ and $\beta = 0$. In this context, we have

$$U_{\rm eff} - \xi = M_0\left[1 + \frac{g}{2}(x^2 + y^2)\right]V(x, y) - EM_0 - \frac{EM_0 g}{2}(x^2 + y^2) + \frac{e^2 B_0^2}{8}(x^2 + y^2). \quad (54)$$

We could choose the anisotropic potential like the one appearing in equation (28) but, since we are working with the symmetric gauge, and we want exact solutions, we shall restrict ourselves to the isotropic case, where

$$U_{\rm eff} = \frac{1}{2}M_0\left(\omega^2 - Eg + \frac{e^2 B_0^2}{4M_0}\right)(x^2 + y^2), \qquad \xi = EM_0, \quad (55)$$

and we can write the above differential equation as

$$-\frac{\hbar^2}{2M_0}\nabla^2\chi + \frac{i\hbar B_0 e}{2M_0}(x\partial_y - y\partial_x)\chi + \frac{1}{M_0}U_{\rm eff}\chi = E\chi. \quad (56)$$

At this point, two alternative routes can be followed in order to get the exact solutions. One can work in polar coordinates, or remain in Cartesian ones and perform some time-dependent rotation transformations [49]. Here, we decided to take this second route and, in order to implement it, we begin by noting that, using $\sigma = e^{-(i/\hbar)Et}\chi(x, y)$, we can work with the corresponding time-dependent Schrödinger equation,

$$-\frac{\hbar^2}{2M_0}\nabla^2\sigma + \frac{i\hbar B_0 e}{2M_0}(x\partial_y - y\partial_x)\sigma + \frac{1}{M_0}U_{\rm eff}\sigma = i\hbar\frac{\partial\sigma}{\partial t}. \quad (57)$$

Then, as we asserted, there is a procedure which can be used in order to decouple equation (57) via variables transformations [49]. It consists in writing a time-dependent rotation matrix like

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \cos\alpha(t) & \sin\alpha(t) \\ -\sin\alpha(t) & \cos\alpha(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad (58)$$

where $t = T$. As a consequence, the momentum operators become

$$p_x = -\frac{\hbar}{i}\frac{\partial}{\partial x} = \frac{\hbar}{i}\left(\cos\alpha(t)\frac{\partial}{\partial x_1} - \sin\alpha(t)\frac{\partial}{\partial y_1}\right),$$
$$p_y = -\frac{\hbar}{i}\frac{\partial}{\partial y} = \frac{\hbar}{i}\left(\sin\alpha(t)\frac{\partial}{\partial x_1} + \cos\alpha(t)\frac{\partial}{\partial y_1}\right).$$

Thus, both the quadratic sum of the spatial coordinates as the momenta, are covariant under these transformations. Furthermore, it can be verified that

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}, \quad (59)$$

so that, it is also covariant. Since we are dealing with a time-dependent rotation, we still have to remember that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T} + \dot{\alpha}\left(y_1\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial y_1}\right) = \frac{\partial}{\partial T} - \dot{\alpha}\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right). \quad (60)$$

Performing the above-described transformations in equation (57), one is left with

$$-\frac{\hbar^2}{2M_0}\nabla^2\sigma + \frac{i\hbar B_0 e}{2M_0}\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right)\sigma + U_{\text{eff}}\sigma = i\hbar\frac{\partial\sigma}{\partial T} - i\hbar\dot{\alpha}\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right)\sigma. \tag{61}$$

Now, defining $\alpha(t)$ such that the crossed term in the variables vanishes, we get

$$\dot{\alpha} = -\frac{eB_0}{2M_0}, \qquad \alpha(T) = -\frac{eB_0}{2M_0}T + c, \tag{62}$$

where c is an integration constant. Hence, we finish with the equation of a two-dimensional isotropic harmonic oscillator,

$$-\frac{\hbar^2}{2M_0}\nabla^2\sigma + U_{\text{eff}}\sigma = i\hbar\frac{\partial\sigma}{\partial T}. \tag{63}$$

So, the energy spectrum of the model is written as

$$EM_0 = \hbar\sqrt{(\Omega^2 - Eg)M_0}(n + m + 1), \tag{64}$$

with $\Omega^2 \equiv \omega^2 + \frac{e^2B_0}{4M_0}$, and we must impose that $Eg < \Omega^2$ in order to keep the energy spectrum real. In this case, as one can see, the system will obey equation (33) with ω replaced by Ω. Thus, one can verify that the case with constant mass is obtained when one takes the limit where $g = 0$, recovering the expected spectrum of the harmonic oscillator in an uniform magnetic.

Finally, we perform the usual separation of variables where

$$\sigma(x_1, y_1, T) = \exp\left(-\frac{i}{\hbar}ET\right)\chi(x_1, y_1), \tag{65}$$

and ends up with an equation for the stationary states given by

$$-\frac{\hbar^2}{2M_0}\nabla^2\chi + U_{\text{eff}}\chi = E\chi, \tag{66}$$

where $\chi = \chi(x_1, y_1)$. Allowing us to write the wavefunctions, which will have the form appearing in equation (36), with the coordinates changed through the above-defined transformation.

4. Final comments

In this work, we presented a general construction of a class of two-dimensional PDM systems in Cartesian coordinates, analyzing an exactly solvable case and discussing its ordering ambiguity and some of their properties. In particular, we constructed its $SU(2)$ coherent state, verifying that it corresponds to a stationary state where the highest probability stays over a circle. Then, we included the interaction with a static magnetic field in the Coulomb gauge and, finally, particularized it to a uniform field in the symmetric gauge, in order to work with an exactly solvable problem. One interesting feature of this system is that the PDM system retains an infinite set of quantum states for a finite-depth bounding potential, which usually do not happens in the case of the constant mass systems.

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